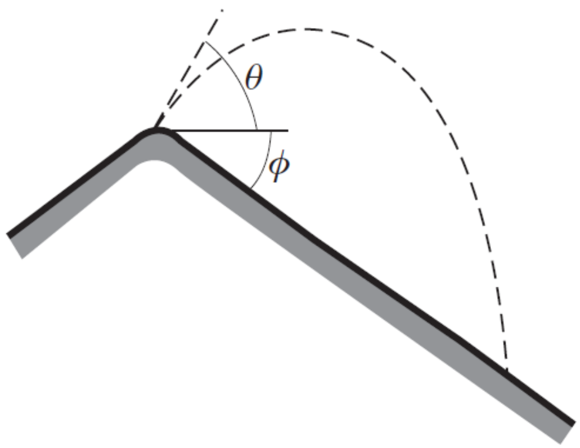
Problem 1.26

Range on a hill
An athlete stands at the peak of a hill that slopes downward uniformly at angle ϕ. At what angle θ from the horizontal should they throw a rock so that it has the greatest range?



Solution

The strategy here is to find an expression for the range, which is defined as the ground distance from where the rock is thrown to where it lands. Once we have this, we will take the derivative of it with respect to θ and then find the value of θ that makes the derivative equal to 0 to maximize the range.

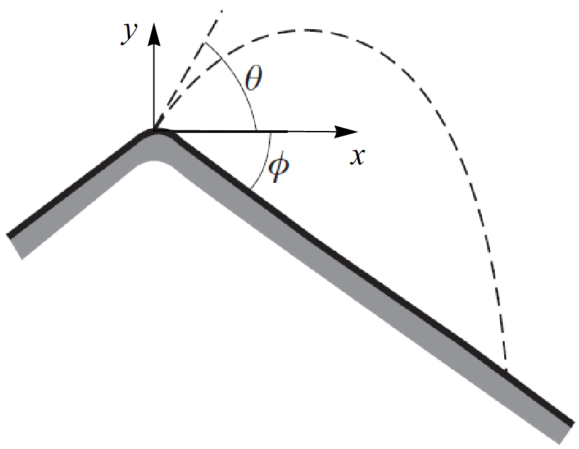


Figure 1: For this problem the origin is chosen to be where the rock is thrown.

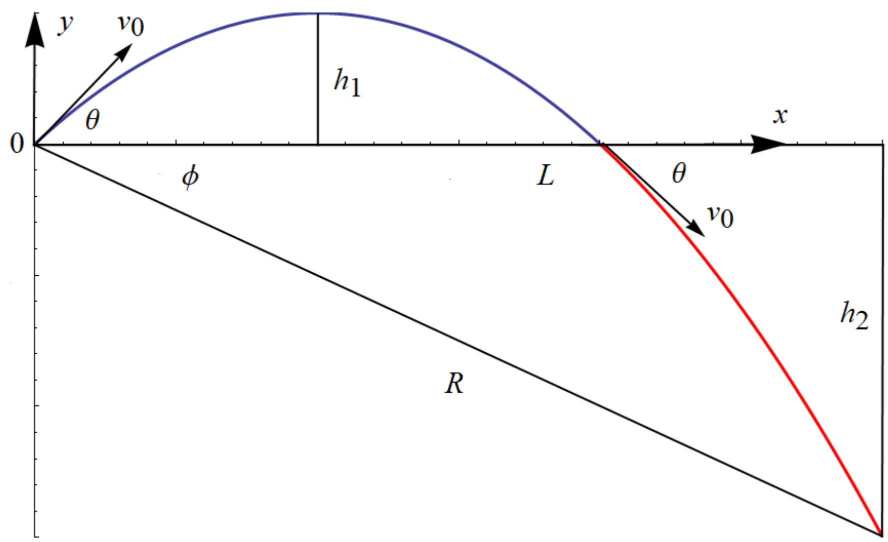


Figure 2: Schematic of the rock's motion with variables to be determined.

R is the variable we're interested in and is the hypotenuse of the right triangle in the figure. From this triangle, the following relationships can be deduced:

$$\cos\phi = \frac{L}{R} \qquad \rightarrow \qquad R\cos\phi = L \tag{1}$$

$$\sin\phi = \frac{h_2}{R} \qquad \rightarrow \qquad R\sin\phi = h_2. \tag{2}$$

L is the horizontal leg of this triangle and is what we will calculate using the kinematic formula,

$$x = x_0 + v_0 t + \frac{1}{2}at^2. \tag{3}$$

Air resistance is neglected, so the acceleration in the x-direction is zero. For the initial velocity the component of v_0 in the x-direction, $v_0\cos\theta$, is used. As for the time, let t_1 denote the time it takes to go from the initial position to the height h_1 and let t_2 denote the time it takes to go from the start of the red curve to the height $-h_2$. Then L can be written as

$$\begin{aligned} L &= v_0(\cos\theta)(t_1 + t_1 + t_2) \\ &= v_0(\cos\theta)(2t_1 + t_2). \end{aligned} \tag{4}$$

t_1 is included twice since it takes as much time for the rock to go up as it does to come down. Because the rock comes to rest at the height h_1, t_1 can be calculated from the kinematic formula, $v = v_0 + at$.

$$0 = v_0\sin\theta + (-g)t_1 \quad \rightarrow \quad gt_1 = v_0\sin\theta \quad \rightarrow \quad t_1 = \frac{v_0\sin\theta}{g}$$

t_2 will be calculated by applying equation (3) in the y-direction. At the start of the red curve, the velocity is v_0 at an angle of θ below the horizontal. The acceleration in the y-direction is due to gravity and is denoted as g.

$$\begin{aligned} -h_2 &= v_0 \sin(-\theta) t_2 + \frac{1}{2}(-g) t_2^2 \\ -h_2 &= -v_0 (\sin\theta) t_2 - \frac{1}{2} g t_2^2 \\ h_2 &= v_0 (\sin\theta) t_2 + \frac{1}{2} g t_2^2 \end{aligned}$$

Solve equation (4) for t_2 and use the result for t_1

$$L = v_0(\cos\theta)(2t_1 + t_2) \quad \rightarrow \quad \frac{L}{v_0 \cos\theta} = 2t_1 + t_2 \quad \rightarrow \quad t_2 = \frac{L}{v_0\cos\theta} - 2t_1 = \frac{L}{v_0\cos\theta} - \frac{2v_0 \sin\theta}{g}$$

and substitute this result into the equation for h_2.

$$h_2 = v_0(\sin\theta)\left(\frac{L}{v_0\cos\theta} - \frac{2v_0\sin\theta}{g}\right) + \frac{1}{2} g \left(\frac{L}{v_0\cos\theta} - \frac{2v_0\sin\theta}{g}\right)^2$$

Substitute equations (1) and (2) for L and h_2, respectively, to make this into an equation we can solve for R.

$$R\sin\phi = v_0(\sin\theta)\left(\frac{R\cos\phi}{v_0\cos\theta} - \frac{2v_0\sin\theta}{g}\right) + \frac{1}{2} g \left(\frac{R\cos\phi}{v_0\cos\theta} - \frac{2v_0\sin\theta}{g}\right)^2$$

Expand the terms on the right side.

$$R\sin\phi = v_0 \frac{R\sin\theta\cos\phi}{v_0\cos\theta} - \frac{2v_0^2\sin^2\theta}{g} + \frac{gR^2\cos^2\phi}{2v_0^2\cos^2\theta} - g\frac{R\cos\phi}{v_0\cos\theta}\cdot\frac{2v_0\sin\theta}{g} + \frac{1}{2} g \frac{4v_0^2\sin^2\theta}{g^2}$$

Simplify the right side.

$$R\sin\phi = \frac{R\sin\theta\cos\phi}{\cos\theta} - \cancel{\frac{2v_0^2\sin^2\theta}{g}} + \frac{gR^2\cos^2\phi}{2v_0^2\cos^2\theta} - \frac{2R\sin\theta\cos\phi}{\cos\theta} + \cancel{\frac{2v_0^2\sin^2\theta}{g}}$$

Cancel the common terms.

$$R\sin\phi = \frac{gR^2\cos^2\phi}{2v_0^2\cos^2\theta} - \frac{R\sin\theta\cos\phi}{\cos\theta}$$

Divide both sides by R ($R \neq 0$) and multiply both sides by $\cos\theta$.

$$\sin\phi\cos\theta = \frac{gR\cos^2\phi}{2v_0^2\cos\theta} - \sin\theta\cos\phi$$

Solve for R.

$$R = \frac{2v_0^2\cos\theta}{g\cos^2\phi}(\sin\phi\cos\theta + \sin\theta\cos\phi)$$

Use the angle addition formula for sine.

$$R = \frac{2v_0^2\cos\theta}{g\cos^2\phi}\sin(\phi+\theta)$$

Now that the range is known, the angle θ will be maximized. This is done by taking the derivative with respect to θ.

$$\begin{aligned}\frac{dR}{d\theta} &= \frac{2v_0^2}{g\cos^2\phi}\frac{d}{d\theta}[\cos\theta\sin(\phi+\theta)] \\ &= \frac{2v_0^2}{g\cos^2\phi}[-\sin\theta\sin(\phi+\theta)+\cos\theta\cos(\phi+\theta)]\end{aligned}$$

The term in square brackets is the angle addition formula for cosine.

$$\begin{aligned} &= \frac{2v_0^2}{g\cos^2\phi}\cos(\phi+\theta+\theta) \\ &= \frac{2v_0^2}{g\cos^2\phi}\cos(\phi+2\theta)\end{aligned}$$

$dR/d\theta$ is equal to zero when

$$\cos(\phi+2\theta)=0, \quad \text{or} \quad \phi+2\theta=\frac{1}{2}(2n+1)\pi, \quad n=0,1,\ldots.$$

As ϕ and θ are acute angles, we choose $n=0$.

$$\phi+2\theta=\frac{\pi}{2}$$

Therefore, the angle (in radians) the athlete should throw the rock to maximize the range is

$$\theta=\frac{1}{2}\left(\frac{\pi}{2}-\phi\right).$$